Exam # 2 Info and Review

Topics

Material covered in class lectures 9/15 to 10/20. (Subgroups and homomorphisms up through Sylow Theorems.)

1. Know what our important theorems say, know how and when to use them:
   - Orbit-Stabilizer theorem
   - the homomorphism theorem aka 1st isomorphism theorem
2. Know results involving divisibility of orders by prime factors:
   - Lagrange's theorem
   - Cauchy's theorem
   - Class equation
   - Sylow Theorems
3. Be able to give precise and correct definitions of the important concepts and examples:
   - dihedral groups
   - kernel of a homomorphism
   - normal subgroup
   - (left/right) coset
   - equivalence relation, equivalence class, partition
   - quotient group
   - direct product, semi-direct product
   - group action, orbit, stabilizer, centralizer
   - Sylow p-subgroup
4. Be able to prove basic results proved in class/HW such as:
   - $\mathrm{Ker}(f) = \{e\}$ if and only if f is injective.
   - If $|G| = p$, where p is a prime, then $G \cong \mathbb{Z}_p$.
   - The operation on G/N is well-defined when N is a normal subgroup of G.
   - If N, A are subgroups of G with N normal, then AN is a subgroup.
   - If A, B are normal subgroups of G such that $A \cap B = \{e\}$ then $\varphi : A \times B \to G$ defined by $\varphi(a, b) = ab$ is a homomorphism.
   - If m, n are relatively prime, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

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Practice problems

Review (easier) homework problems. Here are a few further practice problems.

Question 1 Consider the subgroups of D_5, $H = \langle \rho \rangle$ and $K = \langle \tau \rangle$. Are these normal subgroups?

Question 2 Let G be a group. Consider the relation on G defined by $g \sim h$ if there is an element $k \in G$ such that $g = khk^{-1}$. Show that $\sim$ is an equivalence relation.

Question 3 Suppose that $d|n$. Show that $f : \mathbb{Z}_n \to \mathbb{Z}_d$ defined by $f([a]_n) = [a]_d$ is a well-defined homomorphisms. What is its kernel? What is $\mathbb{Z}_n/\mathrm{Ker(f)}$?

Question 4 Is $\mathbb{Z}_9 \to \mathbb{Z}_5$, $[a]_9 \mapsto [a]_5$ well-defined?

Question 5 For which a is there a semi-direct product $\mathbb{Z}_5 \rtimes_\alpha \mathbb{Z}_a$ (not isomorphic to the direct product)?

Question 6 Is $SO_n \leq O_n$ normal? If so what is O_n/SO_n? Is $O_n \leq GL_n$ normal? If so, what is GL_n/O_n?

Question 7 Does the formula $aH \cdot bH = abH$ define a group structure on the set of cosets G/H for:

(a) $G = S_3, H = \langle (1\,2) \rangle$

(b) $G = S_3, H = \langle (1\,3\,2) \rangle$

Question 8 How many elements does the subgroup of even permutations in S_n have?

Question 9 Show that $\mathbb{Z}_{10}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_5$.

Question 10 Lagrange's theorem tells us that the only possible orders of elements of G are the divisors of $|G|$. Let d be a divisor of 18. Does $\mathbb{Z}_{18}$ always have an element of order d? Does D_9 always have an element of order d?

Question 11 If G is abelian and p is a prime, what are the possible values for n_p, the number of p-Sylow subgroups?

Question 12 (*This is much harder than an exam problem, but good practice to think about Sylow subgroups.*)
Suppose that G is a group of order 88. Let $H \leq G$ be an 11-Sylow subgroup and K a 2-sylow subgroup. Must H be normal? What about K? (In each case, give an argument for normality or provide a counterexample).